Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal
PhD Admission (2013-14, Even Semester)
Written Test

Date : 16.12.2013 **Time : 1 Hour 30 Minutes**

Each question carries 5 marks. Answer as many questions as possible.

1. Let $A_{n\times n}$ be a real skew-symmetric matrix such that Ax is non-zero for non-zero $x \in \mathbb{R}^n$. Show that
   (a) n is even
   (b) $det(I + A) \geq 1$

2. Find the dimension and a basis of the vector space
   $$S = \{(x_1, x_2, x_3, x_4, x_5)^T \in \mathbb{R}^5 : x_1+x_2+x_3+x_4+x_5 = 0, x_2+2x_3-x_4 = 0, x_1-3x_2-x_3+2x_4 = 0\}.$$

3. Find the general solution of the partial differential equation $x^2u_x + y^2u_y = (x + y)u$.

4. (a) Suppose that some tricks you into believing that $233 \times 577 = 135441$. Use congruences to prove in a flash that this is wrong.
   (b) Find a prime factorization of 2419.

5. Consider the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$ on $X = \{a, b, c, d, e\}$. Determine the set of limit points of $A = \{c, d, e\}$.

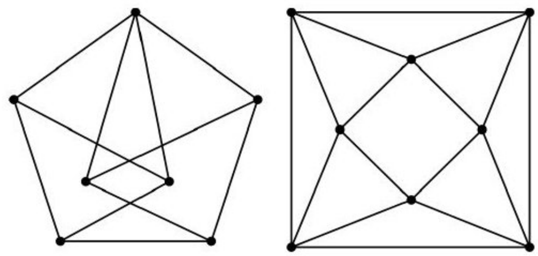
6. Show that the fuction $f(x) = x^2$ is
   (a) uniformly continuous on $[-2, 2]$.
   (b) not uniformly continuous on $[-2, \infty)$.

7. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then show that $f(X)$ is compact.

8. (a) Derive Newton's method for solving a non-linear equation $f(x) = 0$. Also discuss the graphical significance of the method.
   (b) Suppose that the bisection method is started with the interval $[50,\ 60]$. How many steps should be taken to compute a root with relative accuracy is 10^{-12}?

9. Discuss the difference in the two schemes Jacobi and Gauss-Siedel for solving a linear system $Ax = b$. Also check whether the following linear systems converges for these methods with a starting iterate $x = (0,\ 0,\ 0)$.

   (i) $\begin{pmatrix} 5 & 1 & 2 \\ 4 & 7 & 2 \\ 6 & 3 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0.5 & 2 & 1 \\ 0 & 0.1 & 4 \\ 1 & 2 & 0.3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$



10. Find the chromatic numbers of the graphs above. Justify your answer.

11. Give an example of of an infinite group so that each element of which is of finite order.

12. Let $\phi : G \to G'$ be a homomorphism of G onto G'. If G is cyclic show that G' is also cyclic.

13. If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \equiv \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then show that

$$\begin{aligned} P &= (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}BSCA^{-1} \\ Q &= (BD^{-1}C - A)^{-1}BD^{-1} = -PBD^{-1} = -A^{-1}BS \\ R &= (CA^{-1}B - D)^{-1}CA^{-1} = -SCA^{-1} = -D^{-1}CP \\ S &= (D - CA^{-1}B)^{-1} = D^{-1} + D^{-1}CPBD^{-1}. \end{aligned}$$

Mention an application of this result in computational science / engineering.

14. Prove that $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$. Mention an application of this result in computational science / engineering.

15. Consider an image matrix $M_{n\times m}$ and a filtering mask $h_{3\times 3}$. What should be h so that $M \otimes h = M$. Here $\otimes$ denotes a 2-dimensional discrete linear convolution (Assume a zero boundary condition).

16. Consider a Language with the alphabets $\{0, 1\}$ which accepts all the strings with at least one consecutive pair of zeros in it. Design a Deterministic Finite Automation (using the state transition diagram) for the language.

17. An organization has been assigned the network number 150.34.0.0/16 and it needs to create a set of subnets that supports more than 100 and less than 150 hosts on each subnet.

    (a) What is the maximum number of hosts that can be assigned to each subnet?

    (b) What is the maximum number of subnets that can be defined?

    (c) Specify the subnets in dotted decimal notation for the first 6 subnets.

18. The code below shows two processes that must cooperate in computing N^2 by taking the sum of the first N odd integers.

```
           Process P       Process Q
           N = 5;
           Sqr = 0;
loopP:     loopQ:
           if (N==0)       Sqr = Sqr + 2*N + 1;
           goto endP;
           N = N - 1;      goto loopQ;
           goto loopP;
endP:
           print(Sqr);
```

Add appropriate semaphore declarations and signal and wait statements to these programs so that the proper value of Sqr (i.e., 25) will be printed out. Indicate the initial value of every semaphore you add. Insert the semaphore operations so as to preserve the maximum degree of concurrency between the two processes; do not put any non essential constraints on the ordering of operations. Hint: Two semaphores suffice for a simple and elegant solution.

19. A and B cast alternately a pair of dice. A wins if he throws a sum 6 before B throws a sum 7 and B wins if he throws a sum 7 before A throws a sum 6. A begins and the game continues indefinitely. Show that A's chance of winning the game is $\frac{30}{61}$.

20. A particle moves on a circle through points which have been marked $0, 1, 2, 3, 4$(in a clockwise order). At each step it has a probability p of moving to the right(clockwise) and $1 - p$ to the left(counterclockwise). Let X_n denote the location of the particle on the circle after the nth step. The process $\{X_n, n \geq 0\}$ is a Markov chain.

    (a) Find the transition probability matrix.

    (b) Calculate the limiting probabilities.