Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal
PhD Admission (2013-14, Odd Semester)
Written Test

Date : 08.07.2013 **Time : 1 Hour 30 Minutes**

Each question carries 5 marks. Answer as many questions as possible.

1. Let $A = \begin{bmatrix} 8 & 2 \\ 0 & 1 \end{bmatrix}$. Does there exist a matrix B such that $B^3 = A$? Find B, if it exists.

2. Prove or disprove the following statement:
   Let B be a basis of a vector space V. Given that U is a subspace of V, then there exists a subset of B which must be a basis of U.

3. Does there exist solution for the IVP of the form
$$y' = \frac{\cos y}{1 - x^2}, \quad y(0) = y_0$$
   where $|y_0| < \infty$ in $|x| < 1$? Justify.

4. Check whether the following PDE is elliptic, parabolic or hyperbolic :
$$yu_{yy} + 2u_{xy} + xu_{xx} + u_x = 0.$$
   Explain graphically.

5. Let G be a group and ϕ be an automorphism of G. If $a \in G$ is of order $o(a) > 0$, then show that $o(\phi(a)) = o(a)$.

6. Give an example of a set G and a binary operation on G, which is non-associative.

7. Let X be a metric space, $A \subseteq X$. Prove: If A is totally bounded, then A is bounded. Is the converse true? Justify.

8. Give a sequence of functions on $[0, 1]$ to $\mathbb{R}$ which converges pointwise but not uniformly to a continuous function.

9. Let $a, n > 1$ be positive integers. If $a^n - 1$ is prime, prove that one must necessarily have $a = 2$ and n=prime. However, the converse need not be true: Verify this by factorizing $2^{11} - 1$.

10. Let $X = 7^{9999}$.

    (a) Determine the last 3 digits of X if X is in decimal representation.

    (b) Determine the last 8 bits (binary digits) of X if X is in binary representation.

11. Derive the standard five-point formula for the $2D$-Laplace equation, using the method of finte difference.

12. Suppose

$$e^{a\Delta t} = 1 + \frac{\alpha \Delta t}{(\Delta x)^2}\left[e^{ik\Delta x} + e^{-ik\Delta x} - 2\right]$$

where α, a, k are constants and $\Delta t, \Delta x$ are the increments in variables t and x. If $|e^{a\Delta t}| \leq 1$ prove that $\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$.

13. What is the expectation of

    (a) the sum of the points on 'n' dice?

    (b) the product of the points on 'n' dice?

14. From statistical data, it was found that the vowels and consonants in a Samoan word form a Markov chain. A consonant is never followed by a consonant and a vowel has a probaility of 0.51 being followed by a vowel.

    (a) Find the TPM.

    (b) If the first letter of samoan word is a consonant, what is the probaility that

        i. the third letter is a consonant?

        ii. the third letter is a consonant and the fourth is a vowel?

15. If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \equiv \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then show that

$$\begin{aligned}
P &= (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}BSCA^{-1} \\
Q &= (BD^{-1}C - A)^{-1}BD^{-1} = -PBD^{-1} = -A^{-1}BS \\
R &= (CA^{-1}B - D)^{-1}CA^{-1} = -SCA^{-1} = -D^{-1}CP \\
S &= (D - CA^{-1}B)^{-1} = D^{-1} + D^{-1}CPBD^{-1}.
\end{aligned}$$

Mention an application of this result in computational science / engineering.

16. Prove that $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$. Mention an application of this result in computational science / engineering.

17. Prove that a bipartite graph has a unique bipartition (except for interchanging the two partite sets) if and only if it is connected.

18. Prove or disprove :

    (a) Every Eulerian bipartite graph has even number of edges.

    (b) Every Eulerian simple graph with even number of vertices has an even number of edges.

19. Say true or false with justification : Every proper subspace of a normed space is nowhere dense.

20. (a) State Hahn-Banach extension theorem.

    (b) The extension is not unique, in general. Give an example.

    (c) When is the extension unique?

    (d) Prove that the set of all Hahn-Banach extensions is a convex set in the dual of the Banach space.

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